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Spline Finite Strip Bending Analysis of Functionally Graded Plate using Power-law Function

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Abstract—Functionally Graded Materials (FGMs) are widely used in many structural applications because of their high performance of heat resistant and also due to the superior properties they posseess compared to homogeneous material composed of similar constituents. Closed form solutions are already available for the analysis of plated FGM structures for simple loading and boundary conditions. Hence here the authors have developed an approximate solution for FGM plates using Spline Finite Strip Method (SFSM) which can be extended to complicated boundary conditions and loading. Power law idealization is used to show the variation of Young's moduli along the thickness direction. The deflections are obtained using Classical Plate Theory.

Index Terms— Classical Plate Theory, Exponential function, Functionally Graded Material, Power-law function, Sigmoid function, Spline Finite Strip Method, B_3 splines

1 INTRODUCTION

Functionally graded materials (FGM) are the advanced materials in the family of engineering composites made of two or more constituent phases with continuous and smoothly varying composition. These advanced materials with engineering gradients of composition, structure and/or specific properties in the preferred direction/orientation are superior to homogeneous material composed of similar constituents [1]. Functionally graded materials (FGMs) are widely used in many structural applications such as aerospace, nuclear, civil and automotive because of their high performance of heat resistant. The concept of FGM, initially developed for super heat resistant materials to be used in space planes or nuclear fusion reactors, is now of interest to designers of functional materials for energy conversion, dental and orthopaedic implants, sensors and thermo-generators and wear resistant coatings. FGMs are also used for joining dissimilar materials.

An FGM can be prepared by continuously changing the constituents of multi-phase materials in a pre-determined volume fraction of the constituent material. Due to the continuous change in material properties of an FGM, the interfaces between two materials disappear but the characteristics of two or more materials of the composite are preserved. Subsequently the stress singularity at the interface of a composite can be eliminated and thus the bonding strength is enhanced [2], [3]. Because of the wide material variations and applications of FGMs, many research works have already been done for the bending and buckling analysis.

In this study, the Spline Finite Strip bending analysis of PFGM using the power-law idealisation technique is investigated using Classical Plate Theory. The material properties are varied continuously in the thickness direction according to a power-law distribution. The results are compared with the results of the closed form solution developed from Fourier Series Expansion by Chi & Chung [2].

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2 FUNCTIONALLY GRADED MATERIALS

2.1 General

FGMs are microscopically non-homogenous materials in which the mechanical properties vary smoothly and continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. Typically these materials are made from a mixture of ceramics and metal or a combination of different metals. The ceramic constituent of the material provides the high-performance resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture caused by stresses due to high temperature gradient in a very short period of time. They are now being regarded as one of the most promising candidates for future intelligent composites in many engineering structures.

2.2 Mathematical Idealisation Techniques of FGM

Although FGMs are highly heterogeneous, it will be very useful to idealize them as continua with their mechanical properties changing smoothly with respect to the spatial coordinates. The homogenization schemes are necessary to simplify their complicated heterogeneous microstructures inorder to analyse FGMs in an efficient manner. A typical FGM represents a particulate composite with a prescribed distribution of volume fractions of constituent phases. The material properties are generally assumed to follow gradation throughout the thickness in a continuous manner.The Poisson's ratios of the FGM plates are assumed to be constant, but their Young's moduli vary continuously throughout the thickness direction according to the volume fraction of constituents defined by power-law (PFGM), sigmoid (SFGM), or exponential function (EFGM). Power-law and exponential functions are commonly used to describe the variations of material properties of FGMs. However, in both power-law and exponential functions, the stress concentrations appear in one of the interfaces in which the material is continuous but rapidly changing.These three types of variations/gradations are popular for the analysis of material properties.

3 SPLINE FINITE STRIP METHOD

Although closed form analytical method may be possible in

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simple cases of idealized structure and loading, various numerical approaches like FEM, CFSM, SFSM etc. are usually resorted to complex systems and loading conditions. The Finite Element Method (FEM) has been extensively used for the analysis of plated structures. The computational requirement of FEM, in terms of storage space and time is very high, especially in linear prismatic members wherein some of the elements have small width. Hence, this method has only limited application in the stability and non-linear analysis of linear prismatic members modeled using plates and shells, especially when iterative non-linear analysis is needed, as in optimum design. The Classical Finite Strip Method (CFSM), on the other hand, allows more efficient modeling of such prismatic members using strips as elements along the length of the member. This method works well for simple boundary conditions (simply supported, clamped, etc.), but fails to effectively deal with complex boundary conditions and partial and concentrated loads, since the trigonometric functions used to model displacements in the longitudinal direction are infinitely continuous. The continuity and discontinuity requirements can be satisfied by replacing the classical trigonometric function by a spline function as is done in Spline Finite Strip Method (SFSM).

The spline function is defined as a piecewise polynomial of n^{th} degree which is smoothly connected to the adjoining spline functions which has n-1 continuous derivatives. There is variety of splines namely natural spline, cardinal spline, basic spline etc. B_3 spline (cubic basic) is most common and is continuous over only four consecutive sections. Equal and unequal spaced spline series have been used by many researchers to analyse thin and thick plate structures. The unequal splines are more efficient when the structure is subjected to concentrated loads and reactions, when the support of members are either isolated or at irregular locations and when cut-outs are present.

Spline finite strip has all the advantages of classical finite strip and there are additional merits also. The trigonometric series that is used in Classical Finite Strip Method (CFSM) is not the right approximation to model the bending behaviour. Since, the series being infinite in nature the accuracy will depend on number of terms chosen and demands a harmonic analysis. In addition, these series cannot be applied to generalized boundary conditions and loading conditions. The B_3 spline series do not suffer these shortcomings of CFSM. The B_3 spline series is a piecewise cubic polynomial, which is an ideal approximation of the bending behaviour. Another property of B_3 spline is its localized behaviour that makes the stiffness matrix highly banded. Owing to this property, incorporating the boundary conditions is easy, and only three splines adjacent to the constraint need to be modified.

4 PROBLEM FORMULATION

4.1 Power law idealisation

A typical ceramic-metal FGM plate is shown in Fig. 1.The volume fraction of the P-FGM is assumed to obey a power law function:

$$g(z) = \left(\frac{z + h/2}{h}\right)^p \tag{1}$$

where p is the material parameter and h is the thickness of the plate.

Once the local volume fraction $g(z)$ has been defined, the material properties of a P-FGM can be determined by the rule-of-mixture.

$$E(z) = g(z)E_1 + [1 - g(z)]E_2 \tag{2}$$

where E_1 and E_2 are the Young's moduli of the lowest ($z = h/2$) and top surfaces ($z = -h/2$) of the FGM plate, respectively.

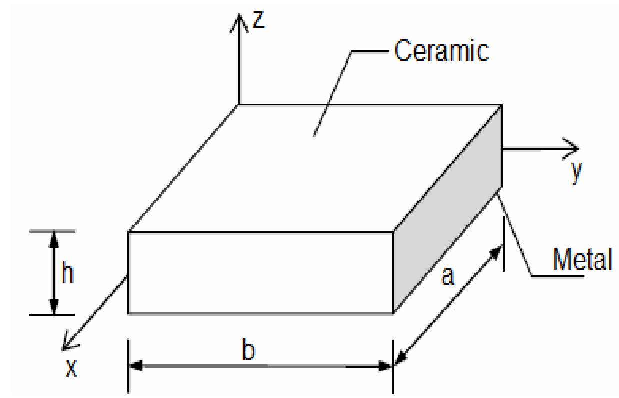

Fig. 1. A typical ceramic metal Functionally Graded Plate.

4.2 Stiffness matrix formulation using classical plate theory

The simplest Equivalent Single Layer (ESL) laminate theory is the Classical Plate Theory which is an extension of Kirchoff's Plate Theory. Here both transverse shear and transverse normal stresses are neglected. The deformationis is entirely due to bending and in-plane stretching. The normal stresses σ_x, σ_y and shear stress τ_{xy} acting in the XY plane are derived. The stress resultants are obtained by integrating stress along the thickness. Thus the axial forces and the bending moments are obtained in terms of coefficients A_{ij}, B_{ij} and C_{ij}.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{matrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{matrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{4}$$

They are the stiffness matrix coefficients which are obtained by the integration of material properties of the FGM plate and are defined as shown:

$$A_{11} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\upsilon(z)^2} dz, \qquad A_{12} = \int_{-h/2}^{h/2} \frac{E(z)\upsilon(z)}{1-\upsilon(z)^2} dz, \tag{5}$$

$$B_{11} = \int_{-h/2}^{h/2} \frac{zE(z)}{1-\upsilon(z)^2} dz, \qquad B_{12} = \int_{-h/2}^{h/2} \frac{zE(z)\upsilon(z)}{1-\upsilon(z)^2} dz,$$

$$C_{11} = \int_{-h/2}^{h/2} \frac{z^2E(z)}{1-\upsilon(z)^2} dz, \qquad C_{12} = \int_{-h/2}^{h/2} \frac{z^2E(z)\upsilon(z)}{1-\upsilon(z)^2} dz,$$

$$A_{66} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\upsilon(z)^2} \left(\frac{1-\upsilon(z)}{2}\right) dz$$

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$$B_{66} = \int_{-h/2}^{h/2} \frac{zE(z)}{1-\upsilon(z)^2}\left(\frac{1-\upsilon(z)}{2}\right)dz$$

$$C_{66} = \int_{-h/2}^{h/2} \frac{z^2E(z)}{1-\upsilon(z)^2}\left(\frac{1-\upsilon(z)}{2}\right)dz$$

The effect of Poisson's ratio in causing deformation of the plate is negligible. Hence the Poisson's ratio is assumed as constant. On substituting (2) in (5), the stiffness matrix coefficients for PFGM plate is obtained as

$$A_{11} = \frac{h}{1-\upsilon^2}\left(\frac{pE_2+E_1}{p+1}\right) \qquad (6)$$

$$B_{11} = \frac{h^2}{1-\upsilon^2}\left(\frac{(E_1-E_2)p}{2(p+1)(p+2)}\right)$$

$$C_{11} = \frac{h^3}{12(1-\upsilon^2)}\left(E_2 - \frac{12(E_1-E_2)}{(p+2)(p+3)}\right)$$

Similarly all the other coefficients can be obtained. These stiffness matrix coefficients are used to derive the equilibrium equations which in turn gives the deflection of the FGM plate subjected to uniformly distributed load.

4 Results

4.1 Square plate subjected to uniformly distributed load

A square FGM plate of width to thickness ratio 50 is considered. The plate is simply supported on its four sides as shown in Fig.2 and is subjected to a uniformly distributed load of 1 kg/cm^2. The Poisson's ratio of the FGM plate is assumed to be constant in the whole plate. υ = 0.3. The Young's modulus at the bottom surface of the FGM plate, E_1 is 2.1x10^6, while that at the top surface of the FGM plate, E_2 varies with the ratio of E_1/E_2. The Young's modulus at any point on the FGM plate varies continuously in the thickness direction based on the volume fraction of the constituents.

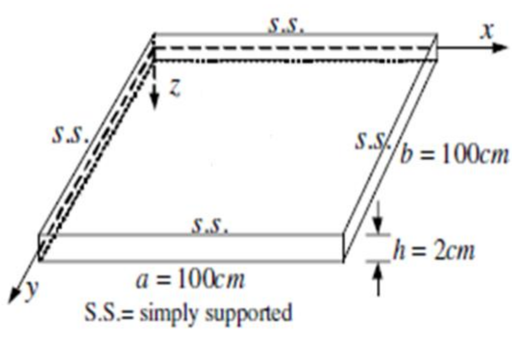

Fig. 2. Configuration of simply supported square FG plate.

Due to the symmetry about the x- and y-axes, only one quarter of the full plate is considered for analysis. In the finite strip method, 5 strips in the longitudinal direction are used to simulate the variation of deflection of the FGM plate.

The SFSM results are obtained for values of E_1=2.1x10^6 kg/cm^2, E_2=0.7x10^6 kg/cm^2, a=b=100cm, h=2cm, υ=0.3, q_0=1 kg/cm^2, E_1/E_2=3 for PFGM. The SFSM results shows good agreement with the closed form solutions given by Chi& Chung (2006) which is shown in Fig. 4 and Fig. 5 for p=0.5 and p=2 respectively.

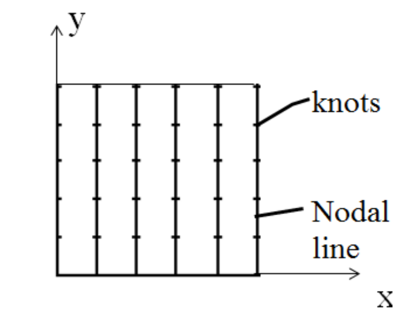

Fig. 3. Discretisation of Functionally Graded Plate.

Fig. 3 shows the discretisatisation of one quarter of the Functionally Graded Plate into different strips. The knots and nodal lines present in the plate are clearly depicted in it.

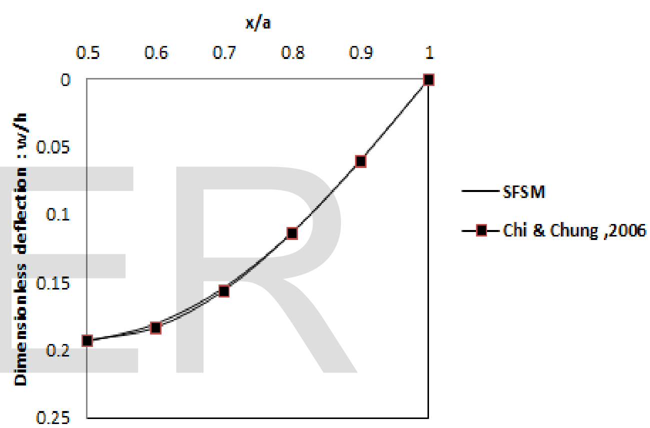

Fig. 4. Dimensionless deflection (w/h) of PFGM plate along the x direction for E_1/E_2=3 & p=0.5.

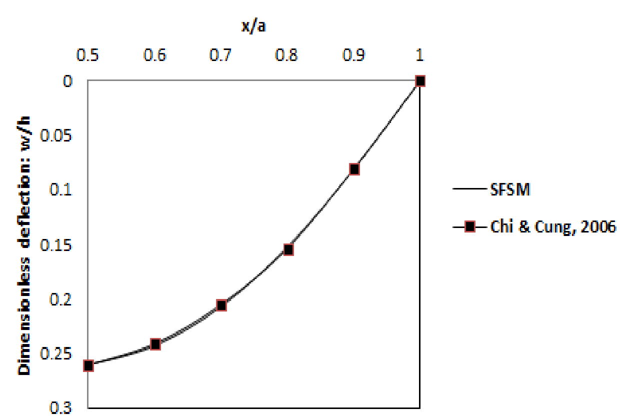

Fig. 5. Dimensionless deflection (w/h) of PFGM plate along the x direction for E_1/E_2=3 & p=2.

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SFSM results obtained for deflection with varying E_1/E_2 ratios for different x/a values for the PFGM plate by assuming the value of material parameter 'p' as 2. Fig. 6 shows the variation of E_1/E_2 values for both theoretical as well as Spline Finite Strip Method. This shows that both the results agree very well with maximum error less than 5%.

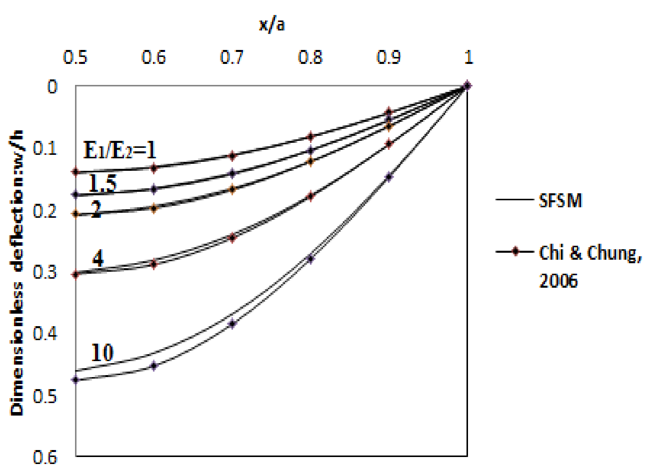

Fig. 6. Dimensionless deflection (w/h) of PFGM plate along the direction for different E_1/E_2

5 Conclusion

The Spline Finite Strip Method results using the power-law function agrees very well with the theoretical result developed for the deflection of the PFGM plate under uniformly distributed load. The results lead to the following conclusions:

1) For values of 'p=0.5', the PFGM shows largest stiffness and gives less deflection. But for other values of 'p' stiffness is found to reduce causing increased deflection.
2) As the value of 'p' increases the stiffness of the plate decreases due to the rapid variation of Young's Modulus towards the lower surface.
3) The more E_1/E_2, the larger deflection 'w', because of less stiffness of the PFGM plate for larger E_1/E_2.

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